5-2 The Chain Rule

Standard 5b: Apply the Chain Rule to differentiate composite functions

Nothing we have done so far could help us take the derivative of this:

$$y = \sqrt{x^2 + 2}$$

But let's try a function we know using a different approach:

$$y = (2x^2 + 1)^2$$

We can FOIL and then use the power rule:

$$y = 4x^4 + 4x^2 + 1$$

$$\frac{dy}{dx} = 16x^3 + 8x$$

or…

$$f(x) = x^2 \qquad g(x) = 2x^2 + 1$$

Replace x with $2x^2 + 1$

A function inside of another function like this is called a *Composite Function*

$$f(g(x)) = (2x^2 + 1)^2$$

We could FOIL and differentiate but what if this composite function were under a radical like the first problem which we have yet to solve?

The Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

The Chain Rule is also known as the "Outside-Inside" rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Derivative of the outside first (f) then the inside (g)

$$f(g(x)) = (2x^2 + 1)^2$$

Apply the Power Rule to the exponent
outside the parentheses

$$\frac{d}{dx} f(g(x)) = 2(2x^2 + 1)^1(4x) = 16x^3 + 8x$$

Look familiar?

Derivative of
the inside term

Now let's try another one:

Consider y to be a composite function that can be broken up

$$y = (x^2 - 2x - 5)^4$$

$$f(x) = x^4 \qquad g(x) = x^2 - 2x - 5$$

Replace x with $x^2 - 2x - 5$

$$f(g(x)) = (x^2 - 2x - 5)^4$$

Apply the Power Rule to the exponent outside the parentheses

$$\frac{d}{dx} f(g(x)) = 4(x^2 - 2x - 5)^3 (2x - 2)$$

Outside-Inside

$$= 4(2x - 2)(x^2 - 2x - 5)^3$$

Now let's try another one:

Consider y to be a composite function that can be broken up

$$y = \frac{1}{(x^2 - 2x - 5)^4}$$

$$f(x) = \frac{1}{x^4} = x^{-4} \qquad g(x) = x^2 - 2x - 5$$

Replace x with $x^2 - 2x$ - 5

$$f(g(x)) = (x^2 - 2x - 5)^{-4}$$

$$\frac{d}{dx} f(g(x)) = -4(x^2 - 2x - 5)^{-5}(2x - 2)$$

Outside-Inside

$$= \frac{-4(2x - 2)}{(x^2 - 2x - 5)^5}$$

$$y = (x^2 - 2)^3$$

$$y = \sqrt{x^2 - 2}$$

$$y' = 6x(x^2 - 2)^2$$

$$y = \sqrt{\frac{3x - 4}{x^2 - 2}}$$

$$y = (x^2 - 2)^3$$

$$y' = 6x(x^2 - 2)^2$$

$$y = \sqrt{x^2 - 2}$$

$$y = (x^2 - 2)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x^2 - 2)^{-\frac{1}{2}}(2x)$$

$$y' = \frac{x}{(x^2 - 2)^{\frac{1}{2}}}$$

$$y' = \frac{x}{\sqrt{x^2 - 2}}$$

$$y = \sqrt{\frac{3x-4}{x^2-2}} \qquad = \left(\frac{3x-4}{x^2-2}\right)^{\frac{1}{2}}$$

Quotient Rule

f' g g' f

$$y' = \frac{1}{2}\left(\frac{3x-4}{x^2-2}\right)^{-\frac{1}{2}}\left(\frac{3(x^2-2)-2x(3x-4)}{(x^2-2)^2}\right)$$

g^2

$$y' = \frac{-3x^2+8x-6}{2(x^2-2)^2}\left(\frac{3x-4}{x^2-2}\right)^{-\frac{1}{2}} = \frac{-3x^2+8x-6}{2(x^2-2)^2}\left(\frac{x^2-2}{3x-4}\right)^{\frac{1}{2}}$$

$$y = \sqrt{\frac{3x-4}{x^2-2}}$$

The Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$y' = \frac{1}{2}\left(\frac{x^2-2}{3x-4}\right)^{\frac{1}{2}} \frac{-3x^2+8x-6}{(x^2-2)^2}$$

$$y' = \frac{-3x^2+8x-6}{2(x^2-2)^2}\sqrt{\frac{x^2-2}{3x-4}}$$

[redacted]