7-5: L' Hôpital' s Rule

Actually, L' Hôpital' s Rule was developed by his teacher Johann Bernoulli. De l' Hôpital paid Bernoulli for private lessons, and then published the first Calculus book based on those lessons.

Guillaume De l'Hôpital
1661 - 1704

7-5: L’ Hôpital’ s Rule

Johann Bernoulli
1667 - 1748

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Consider: $\lim_{x\to 2}\frac{x^2-4}{x-2}$

If we try to evaluate this by direct substitution, we get: $\frac{0}{0}$

Zero divided by zero can not be evaluated, and is an example of **indeterminate form**.

In this case, we can evaluate this limit by factoring and canceling:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

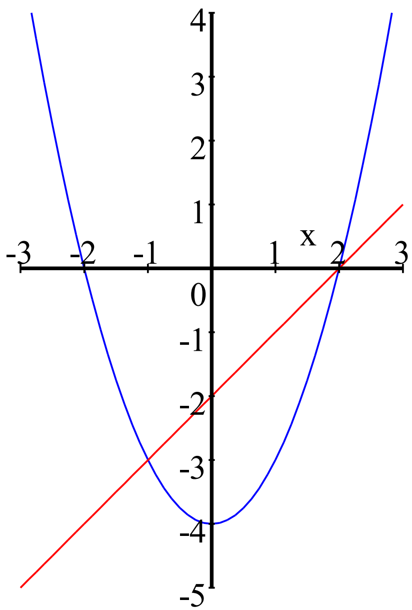
→

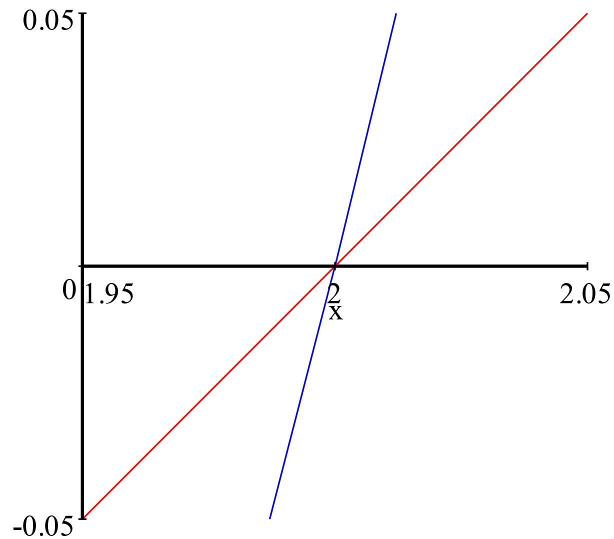
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2}$$

The limit is the ratio of the numerator over the denominator as x approaches 2.

x^2-4

$x-2$





Straight lines.

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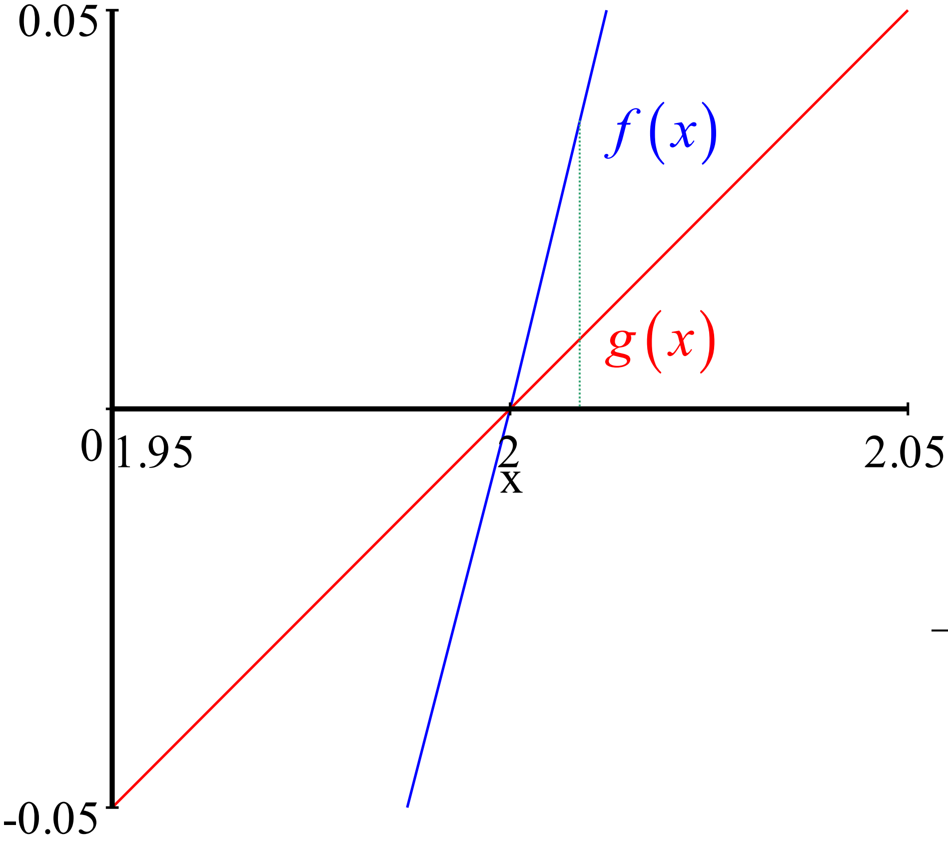
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2}$$

So how can we use this to help us find the limit here?



At $x=2$

$$\frac{f(x)\ \text{approaches its tangent line}}{g(x)\ \text{approaches its tangent line}}$$

$$\frac{f(2)}{g(2)} = \frac{\to f'(2)(x-2)+0}{\to g'(2)(x-2)+0}$$

$$= \frac{f'(2)}{g'(2)}$$

→

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\frac{d}{dx}\left(x^2-4\right)}{\frac{d}{dx}\left(x-2\right)} = \lim_{x\to 2}\frac{2x}{1} = 4$$

L' Hôpital' s Rule:

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

→

Example:

$$\lim_{x\to1}\frac{\ln x}{x-x^2} = \frac{0}{0}$$

First make sure it's indeterminate

Now we use L'Hopital's Rule

$$= \lim_{x\to1}\frac{1/x}{1-2x} \longrightarrow \frac{1}{-1} = -1$$

→

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \quad \longleftarrow \quad \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \quad \longleftarrow \quad \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \quad \longleftarrow \quad \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

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