End Behavior Models and Asymptotes

Standard 4b: Determine the end behavior of a rational function from a model, polynomial long division, or infinite limits and sketch the horizontal or slant asymptote.

When n < m
(top < bottom)

$$\lim_{x\to\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = 0$$

When n > m
(top > bottom)

$$\lim_{x\to\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty$$

$$\lim_{x\to-\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty \text{ when } n - m \text{ is even}$$

$$\lim_{x\to-\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = -\infty \text{ when } n - m \text{ is odd}$$

When n = m

$$\lim_{x\to\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \frac{a}{b}$$

We'll come back to this one a little later

When n < m
(top < bottom)

$$\lim_{x\to\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = 0$$

When n > m
(top > bottom)

$$\lim_{x\to\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty$$

$$\lim_{x\to-\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty \text{ when n – m is even}$$

$$\lim_{x\to-\infty} \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = -\infty \text{ when n – m is odd}$$

Examples to graph on your calculator

$$y = \frac{2x^2 + x - 1}{x + 2} \qquad y = \frac{x^3 - 5x + 1}{x - 1}$$

When n < m
(top < bottom)

$$\lim_{x\to\infty}\frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = 0$$

When n > m
(top > bottom)

$$\lim_{x\to\infty}\frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty$$

$$\lim_{x\to-\infty}\frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = \infty$$

$$\lim_{x\to-\infty}\frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots} = -\infty$$

So how does this all relate to asymptotes and End Behavior Models?

These criteria can be found throughout Section 4-2

End Behavior Model (EBM):

For $y=\dfrac{A_n x^n + A_{n-1}x^{n-1}\ldots}{B_m x^m + B_{m-1}x^{m-1}\ldots}$, the EBM is $y=\dfrac{A_n x^n}{B_m x^m}$.

General Rule for End Behavior Asymptotes:

For $y=\dfrac{A_n x^n + A_{n-1}x^{n-1}\ldots}{B_m x^m + B_{m-1}x^{m-1}\ldots}$,

if $n<m$, there is a horizontal asymptote and it is $y=0$;

if $n=m$, there is a horizontal asymptote and it is $y=\dfrac{A_n}{B_m}$;

if $n=m+1$, there is a slant asymptote;

if $n>m+1$, there is an end behavior model rather than an asymptote.

General Rule for Horizontal Asymptotes:

For $y=\dfrac{A_n x^n + A_{n-1}x^{n-1}\ldots}{B_m x^m + B_{m-1}x^{m-1}\ldots}$,

if $n<m$, there is a horizontal asymptote and it is $y=0$;

if $n=m$, there is a horizontal asymptote and it is $y=\dfrac{A_n}{B_m}$;

and if $n>m$, there is no horizontal asymptote.

General Rule for Slant Asymptotes:

For $y=\dfrac{A_n x^n + A_{n-1}x^{n-1}\ldots}{B_m x^m + B_{m-1}x^{m-1}\ldots}$, if $n=m+1$, there is a slant asymptote.

With Rational Functions, End Behavior Models are determined by infinite limits

$$y = \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots}$$

End Behavior Model (EBM) for y is:

$$y = \frac{ax^n}{bx^m}$$

As long as n ≤ m (top less than bottom), y will have a horizontal asymptote. The criteria for horizontal asymptotes are on pg 198

$$y = \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots}$$

End Behavior Model for y is: $y = \frac{ax^n}{bx^m}$

$$y = \frac{4x^2 + 7x - 6}{2x^2 - 11x + 5}$$

End Behavior Model here is: $y = \frac{4x^2}{2x^2} = \frac{4}{2} = 2$

So this function has a horizontal asymptote at y = 2

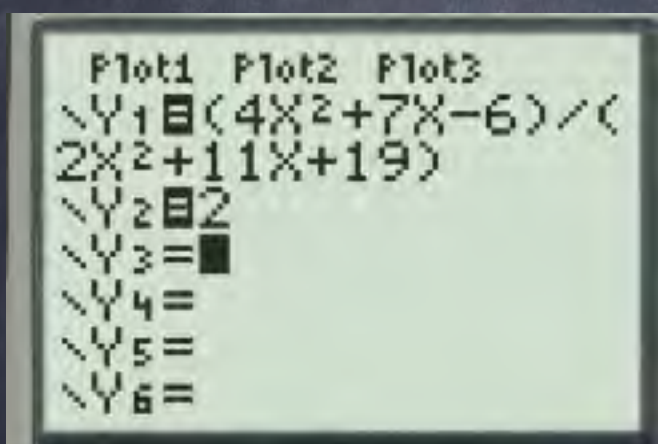
Here's what it will look like on the graphing calculator

$$y = \frac{ax^n + cx^{n-1} + \ldots}{bx^m + dx^{m-1} + \ldots}$$

End Behavior Model for y is: $y = \frac{ax^n}{bx^m}$

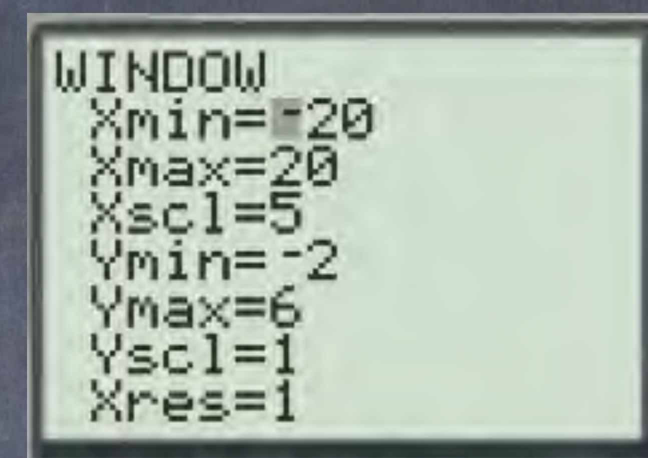
$$y = \frac{4x^2 + 7x - 6}{2x^2 - 11x + 5}$$

So this function has a horizontal asymptote at y = 2



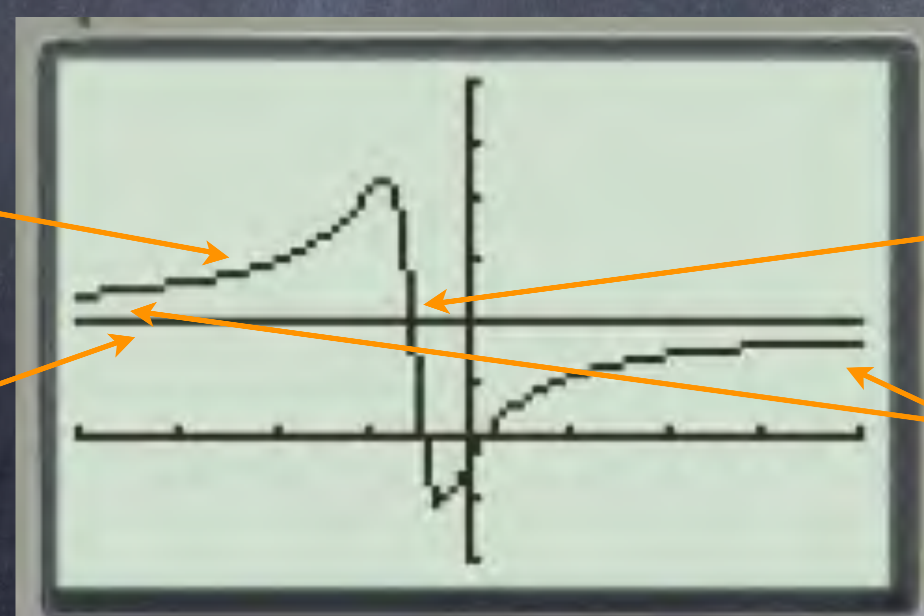
Remember to use parentheses as necessary

Choose your window settings as needed to show the end behavior



Here's the function...

Here's y = 2



It's ok for a function to cross the horizontal asymptote even more than once. As long as the function keeps approaching the asymptote as x goes to ±∞

But if n is greater than m by 1 (n = m + 1), y will have a slant asymptote. Here is where long division comes in.

$$y = \frac{x^4 + 3x^3 - x + 4}{2x^3 + 4x^2}$$

Long Division

$$\frac{1}{2}x + \frac{1}{2} \quad + \frac{-2x^2 - x + 4}{2x^3 + 4x^2}$$

$$2x^3 + 4x^2 \enclose{longdiv}{x^4 + 3x^3 - x + 4}$$

$$-\left(x^4 + 2x^3\right)$$

$$x^3 - x$$

$$-\left(x^3 + 2x^2\right)$$

$$-2x^2 - x + 4$$

Once the degree here is smaller than the degree of the divisor, we're done

End Behavior Model (EBM) for y (slant asymptote) is:

$$y = \frac{1}{2}x + \frac{1}{2}$$

No need to worry about the remainder

Graph both the function and the asymptote to see for yourself

But if n is greater than m by 1 (n = m + 1), y will have a slant asymptote. Here is where long division comes in.

$$y = \frac{2x^2 + x - 1}{x + 2}$$

Long Division

In this case you could use synthetic division here but only because the divisor has a degree of 1

$$\begin{array}{r|l} & 2x - 3 \\ x+2 & 2x^2 + x - 1 \\ & -(2x^2 + 4x) \\ & \quad -3x - 1 \\ & \quad -(-3x - 6) \\ & \qquad 5 \end{array} \Rightarrow \underline{2x - 3} + \frac{5}{x+2}$$

End Behavior Model (EBM) for y (slant asymptote) is:

$$y = 2x - 3$$

No need to worry about the remainder

Graph both the function and the asymptote to see for yourself